STUDY OF THE COLLAPSE OF A CAVITATION CAVITY CLOSE TO A SOLID WALL

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The dynamics of cavitation cavities are affected by the presence of a wall. An expression is obtained for the pressure fluctuations acting on the wall.

So far the overwhelming majority of studies of the dynamics of cavitation cavities have been based on the assumption that the cavity collapses in an infinite medium. Apparently, the only exception is Khoroshev's theoretical study of "The Effect of a Wall on the Process of Collapse of Cavitation Cavities" [1] in which the influence of the proximity of an absolutely rigid wall on the dynamics of a single cavity is analyzed in the first approximation. However, the result obtained, which indicates a reduction in the pressure at the wall as compared with the case of an infinite medium, is inaccurate, since the motion of the center of the bubble in the direction of the wall during collapse is not taken into account.

The present paper reports the results of a study of the collapse of a single spherical vapor-air cavitation cavity at constant pressure P_0 in the proximity of an absolutely rigid wall in an incompressible liquid with allowance for the motion of the cavity toward the wall.

In order to derive the equations of motion of a cavity close to a solid surface we employ the method of Lagrange. As generalized coordinates we take the radius R and the distance of the cavity from the boundary b.

The applicability of this general method of mechanics to the motion of a sphere in a liquid was considered by Lamb [2]. If K is the expression for the kinetic energy of the liquid, it must satisfy the differential equations

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{R}} - \frac{\partial K}{\partial R} = F_R,$$

$$\frac{d}{dt}\frac{\partial K}{\partial U} - \frac{\partial K}{\partial b} = 0, \qquad (1)$$

where $\bar{F}_R$ is the force acting on the surface of the cavity in the direction of the generalized coordinate R. We assume that the cavity is filled with the vapor of the liquid and a gas, which is compressed adiabatically. Then, for the assumed model of cavity motion, with allowance for the forces of surface tension, the force

$$F_R = 4\pi R^2 \left[P_d - \frac{2\sigma}{R} + Q\left(\frac{R_0}{R}\right)^{3\gamma} \right]. \qquad (2)$$

Since the cavity moves in an incompressible liquid, the flow associated with that motion should be irrotational with velocity potential φ satisfying the solution of the Laplace equation.

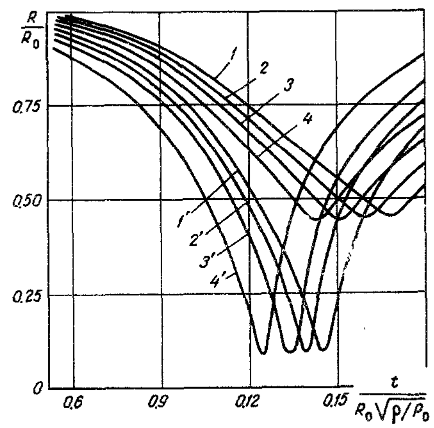

Fig. 1. Radius-time curves for the collapse of a cavitation cavity at a wall (R_0 = 4 mm); 1–4) at Q/P_0 = = 0.2; 1'–4') 0.05; 1, 1') at b_0/R_0= = 20; 2, 2') 2; 3, 3') 1.5; 4, 4') 1.

The kinetic energy can be determined from the values of the velocity potential and its variation at the boundary surfaces by means of the formula [2]

$$K = -\frac{\rho}{2}\iint \varphi \frac{\partial \varphi}{\partial n}\, dS. \qquad (3)$$

In this expression the derivatives of the velocity potential with respect to the normal $\partial\varphi/\partial n$ must satisfy the boundary conditions at the surfaces of the cavity and the rigid wall. In order for the flow near the wall to be parallel to the latter, the condition $\partial\varphi/\partial n = 0$ (impermeability condition) must be satisfied.

The limitations imposed on the flow by the presence of geometric boundaries can be satisfied by superimposing on the flow a suitable combination of sources and dipoles.

The total potential of the combination of sources and dipoles will characterize the motion of the gas cavity near the solid surface. The complexity of the solution of the problem of motion of a cavitation cavity near a solid wall is a consequence of the difficulties associated with the simultaneous satisfaction of the boundary conditions at the rigid wall and the surface of the cavity.

The impermeability condition can be satisfied by using the method of images, according to which the required velocity distribution along the wall can be ensured by replacing it with imaginary sources located at points coinciding with the mirror images of the real sources behind the wall.

In the method of images it is usually assumed that a radially moving cavity is equivalent to a point source or sink at its center. However, the flow created by

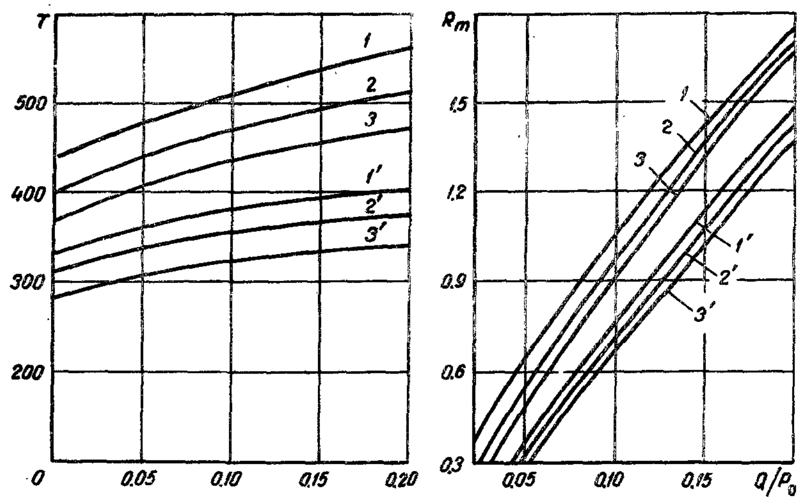

Fig. 2. Collapse time T (μsec) and least radius R_m (mm) as functions of the gas content of the cavity Q/P_0 for a cavity collapsing at a wall: 1–3) at R_0 = = 4 mm; 1'–3') 3 mm; 1, 1') at b_0/R_0 = 1.1; 2, 2') 2; 3, 3') 20.

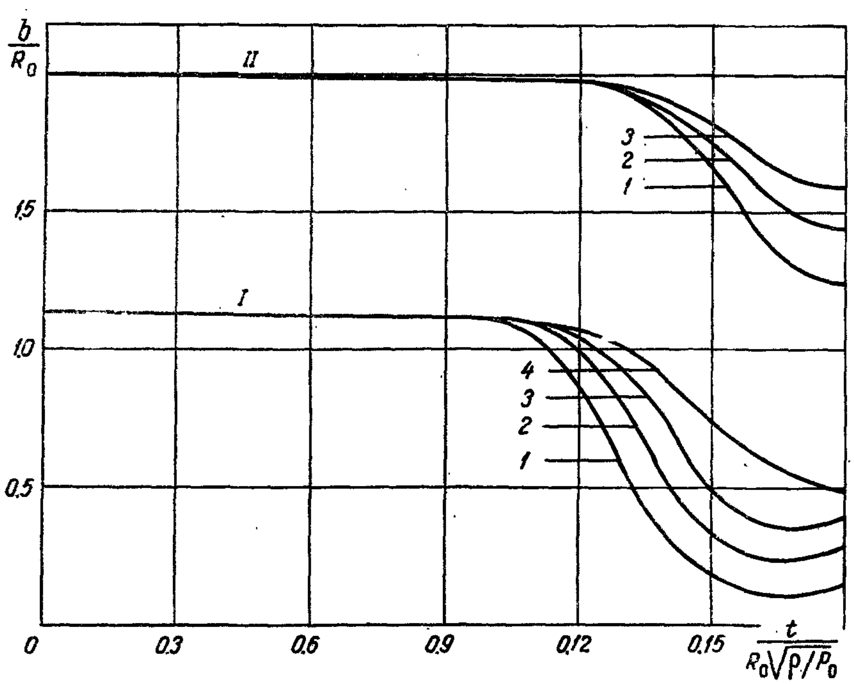

Fig. 3. Displacement of cavitation cavity upon collapse at a wall (R_0 = 4 mm) for b_0/R_0 = 1.1 (I) and 2 (II); 1) at Q/P_0 = 0.05; 2) 0.08; 3) 0.1; 4) 0.2.

the two sources (real and imaginary) cannot be considered real, since the imaginary source will create a flow across the boundary of the cavity.

If the sphere is small as compared with its distance from the boundary, this flow will be almost uniform at all points on its surface, and its distribution will be almost the same as the distribution around a sphere that moves away from or approaches the boundary as the bubble contracts or expands. Thus, obviously, the motion of the bubble can create the necessary flow distribution close to the rigid boundary.

The potential of the flow around a spherical cavitation cavity in translational motion is determined by the potential, applied at the center of the sphere, of a dipole with moment oriented along the axis of motion.

Thus, the total potential of the flow around a spherical cavity whose dimensions are small (as compared with its distance from the solid boundary) may be represented in a coordinate system rigidly tied to the cavity in the form

$$\varphi = \varphi_R \frac{dR}{dt} + \varphi_z U = R^2 \frac{dR}{dt} \Big/ r + R^2 \frac{dR}{dt} \Big/ \sqrt{r^2 + 4b^2 - 4br\cos\Theta} + \frac{1}{2}\frac{UR^3\cos\Theta}{r^2} + \frac{1}{2} UR^3 \sqrt{1 - \sin^2\Theta \frac{r^2}{r^2 + 4b^2 - 4br\cos\Theta}} \times (r^2 + 4b^2 - 4br\cos\Theta), \tag{4}$$

where φ_Z and φ_R are the velocity potentials for unit velocity of translational motion and pulsation.

In this expression the first and third terms determine the potential of the source and the dipole applied at the center of the sphere and the second and fourth terms their reflections in the solid wall.

The derivative $\partial\varphi/\partial n$ must satisfy the boundary conditions on the sphere and at the wall. These conditions may be written in the form

$$\text{Sphere}\left\{\begin{matrix} \dfrac{\partial\varphi_R}{\partial n} = -1 & \dfrac{\partial\varphi_R}{\partial n} = 0 \\ \dfrac{\partial\varphi_z}{\partial n} = -\cos\Theta & \dfrac{\partial\varphi_z}{\partial n} = 0 \end{matrix}\right\}\text{Plane.} \tag{5}$$

If these conditions are applied to the equation for the kinetic energy, the integrals over the plane disappear; as a result we have the expression

$$K = -\frac{\rho}{2}\left[\left(\frac{dR}{dt}\right)^2 \iint \varphi_R\, dS + 2U\frac{dR}{dt}\iint \varphi_z\, dS + U^2 \iint \varphi_z \cos\Theta\, dS\right], \tag{6}$$

in which the integrals must be evaluated over the surface of the sphere. In deriving this expression we made use of the equation

$$\iint \left(\varphi_R \frac{\partial\varphi_z}{\partial n} - \varphi_z \frac{\partial\varphi_R}{\partial n}\right) dS = 0,$$

which is valid for any functions φ_R, φ_Z that are a solution of the Laplace equation. Thus, in order to calculate the total kinetic energy it is necessary only to determine the mean values of φ_R, φ_Z and $\varphi_Z \cos\Theta$ over the sphere. After the mean values of φ_R, φ_Z and $\varphi_Z \cos\Theta$ over the sphere have been calculated, the expression for the kinetic energy takes the form

$$K = 2\pi R^3 \rho\left[1 + \frac{R}{2b}\right]\left(\frac{dR}{dt}\right)^2 + \frac{1}{3}\pi R^3 \rho\left[1 + \frac{3R^3}{8b^3}\right]U^2. \tag{7}$$

Substitution of the expressions for K and F_R in Eq. (1) leads to a system of differential equations for the radial and translational motions of a spherical vapor-air cavitation cavity near a rigid wall

$$\frac{d^2R}{dt^2} = \frac{2b}{R(R+2b)\rho}\left[P_d - \frac{2\sigma}{R} - P_0 + Q\left(\frac{R_0}{R}\right)^{3\gamma}\right] - \frac{3b+2R}{R(R+2b)}\left(\frac{dR}{dt}\right)^2 + \frac{b}{2R(R+2b)}U^2 + \frac{R}{(R+2b)b}\frac{dR}{dt}U,$$

$$\frac{d^2 b}{dt^2} = -\frac{3}{2}\frac{R}{b^2}\left(\frac{dR}{dt}\right)^2 - \frac{3}{R}\frac{dR}{dt}U. \tag{8}$$

As $b \to \infty$ the first of Eqs. (8) reduces to the familiar formula for the acceleration of a collapsing vapor-air cavity in an infinite medium [4]

$$\ddot{R} = -\frac{3}{2}\frac{\dot{R}^2}{R} + \frac{[P_d - (2\sigma/R) - P_0 + Q(R_0/R)^{3\gamma}]}{\rho R}. \tag{9}$$

If the dynamics of motion of the bubble are known, i. e., its radius and displacement as a function of time, the pressure distribution in the surrounding liquid can be found using the Lagrange-Cauchy equation

$$\frac{P}{\rho} = -\frac{1}{2}(\text{grad}\,\varphi)^2 + \frac{\partial\varphi}{\partial t} + \frac{P_0}{\rho}. \tag{10}$$

In using this equation it is necessary to keep in mind that $\partial\varphi/\partial t$ should be differentiated for a point whose position in space is fixed. However, for convenience, the velocity potential has been expressed in coordinates moving at the velocity U of the center of the sphere. Therefore, a point fixed in space possesses the velocity $-U$. Thus the expression for the derivative $\partial\varphi/\partial t$ takes the form

$$\frac{\partial\varphi}{\partial t} = \left(\frac{\partial\varphi}{\partial t}\right)_m - U\left(\frac{\partial\varphi}{\partial x}\right)_t, \tag{11}$$

where $(\partial\varphi/\partial t)_m$ relates to fixed positions of the point in the moving coordinate system and the derivative $\partial\varphi/\partial x$ is evaluated in the direction of U.

Eq. (10) takes the form

$$\frac{P}{\rho} = -\frac{1}{2}(\text{grad}\,\varphi)^2 + \left(\frac{\partial\varphi}{\partial t}\right)_m - U\left(\frac{\partial\varphi}{\partial x}\right)_t + \frac{P_0}{\rho}. \tag{12}$$

The derivative $(\partial\varphi/\partial x)_t$ in the equation is given by

$$\left(\frac{\partial\varphi}{\partial x}\right)_t = \cos\Theta\frac{\partial\varphi}{\partial r} - \frac{\sin\Theta}{r}\frac{\partial\varphi}{\partial\Theta}. \tag{13}$$

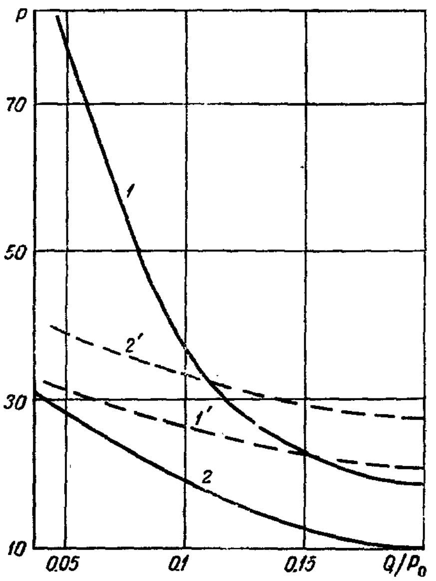

Fig. 4. Pressure at wall P (db rel. 1 atm) as a function of the ratio Q/P_0 for the collapse of a cavitation (1, 2) and the same relation on the assumption that the radial velocities and accelerations of the bubble surface are the same as for the collapse of a bubble in an infinite medium (1', 2') at distances from the wall b_0/R_0 = 1.2 (1, 1') and 2 (2, 2').

Substitution of the derivatives of φ in the equation for the pressure distribution gives the following formula:

$$\begin{aligned}
\frac{P}{\rho} &= \frac{P_0}{\rho} + \frac{2R\dot{R}^2 + R^2\ddot{R}}{r} + \frac{2R\dot{R}^2 + R^2\ddot{R}}{M} - \\
&- \frac{1}{2}\frac{R^4}{r^4}\Big[\dot{R} + \frac{R}{r}U\cos\Theta + \frac{\dot{R}r^2(r-2b\cos\Theta)}{M^3} + \\
&+ \frac{RUr^2N(r-2b\cos\Theta)}{M^4} + \frac{RUr^3\sin^2\Theta}{2NM^4} - \\
&- \frac{RUr^4\sin^2\Theta(r-2b\cos\Theta)}{2NM^6}\Big]^2 - \frac{1}{2}\frac{R^2}{r^2}\sin^2\Theta\times \\
&\times\Big[\frac{1}{2}\frac{R^2}{r^2}U + \frac{R\dot{R}2br}{M^3} + \frac{R^2U2brN}{M^4} - \\
&- \frac{R^2Ur^2\sin\Theta\cos\Theta}{2NM^4} - \frac{R^2Ubr^3\sin^2\Theta}{NM^6}\Big]^2 + \frac{1}{2}\frac{R^2}{r^2}\times \\
&\times[5\dot{R}U + R\dot{U}]\cos\Theta + \frac{1}{2}\frac{R^3}{r^3}U^2(2\cos^2\Theta - \sin^2\Theta) + \\
&+ \frac{3R^2\dot{R}U + R^3\dot{U}N}{2M^2} + \\
&+ \frac{R^2\dot{R}U[(r-2b\cos\Theta)\cos\Theta + 2b\sin^2\Theta + 2(2b - r\cos\Theta)]}{M^3} + \\
&+ \frac{R^3U^2N[r + 4b - 2(b+r)\cos\Theta - 2b\sin^2\Theta]}{M^4} + \\
&+ R^3U^2[2r^2\sin^2\Theta(r\cos\Theta - 2b) + 2br^2\sin^4\Theta - \\
&- r^2\sin^2\Theta\cos\Theta(r - 2b\cos\Theta)][2NM^6]^{-1}. \qquad (14)
\end{aligned}$$

The complex relation between the pressure and the angle Θ is a result of the spherical asymmetry of the flow due to the presence of a solid wall. It should be noted that this pressure variation is valid only on the artificial assumption that the bubble is spherical in shape.

To obtain a comparative estimate of the pressure variation in the case of collapse of a cavity close to a solid wall and in an infinite medium we will find the value of the pressure in these when $\Theta = 0°$ at a distance b from the cavity.

When $\Theta = 0°$ and r = b, Eq. (14) takes the form

$$\left(\frac{P}{\rho}\right)_A = \frac{P_0}{\rho} + \frac{2[2R\dot{R}^2 + R^2\ddot{R}]}{b} - \frac{R^3\dot{U} + 5R^2\dot{R}U}{b^2} + \frac{2R^3U^2}{b^3}. \qquad (15)$$

In the case of an infinite medium the pressure distribution is given by the known equation

$$\frac{P}{\rho} = \frac{2R\dot{R} + R^2\ddot{R}}{r} - \frac{1}{2}\frac{R^4\dot{R}^2}{r^4} + \frac{P_0}{\rho}. \qquad (16)$$

If we disregard the pressure component due to translational motion in (15) and the second term of (16), in the first case the pressure at a distance b from the cavity at point A will be twice as great as in the case of collapse in an infinite medium. Here it is assumed that the dynamics of originally identical bubbles are the same in both cases. In reality, in the case of collapse close to a solid surface owing to the presence of translational motion of the cavity and the expenditure of part of the original energy on this motion there is a decrease in radial velocity and a change in the dependence of the cavity radius on time as well as an increase in the collapse period.

By solving the system of equations of motion (8) on a M-20 computer, we were able to estimate the degree of variation of the parameters of bubble motion in the case of collapse at a solid wall.

Figure 1 shows the radius as a function of time for the collapse of a cavity near a wall at different gas contents in the cavity Q/P_0 and different initial distances from the wall.

As $b \to \infty$ the first equation in system (8) goes over into the expression describing the motion of a gas bubble in an infinite medium. Therefore the case b_0/R_0 = = 20 will correspond most closely to the collapse of a cavity in an infinite fluid.

From these curves it can be seen that the initial gas content of the cavity and its distance from the wall have an important influence on the period and minimum radius of collapse (Fig. 2) as well as on the motion of the bubble (Fig. 3). These parameters will have an even greater effect on the maximum pressure induced by the cavity. Using calculated values of the radii, velocities, accelerations, and displacements, we can determine the maximum pressure acting on the wall from Eq. (14). Figure 4 shows this pressure as a function of the ratio Q/P_0 (P_0 is the external pressure assumed equal to 1 atm) for the collapse of a bubble with the initial dimension R_0 = 4 mm.

At other initial dimensions the pressure distribution will be qualitatively similar. The broken-line curves in the same figure represent the maximum pressures acting on the wall on the assumption that the radial velocities and accelerations of the bubble surface are the same as for the collapse of a bubble in an infinite medium.

At $b_0/R_0 = 1.2$ and low gas contents the broken-line curve lies below the curve calculated from Eq. (14). This is attributable to the intense translational motion of the bubble toward the wall at close initial distances and low gas contents. As a result of this motion the last terms in the equation for the pressure (14) assume large values and determine the maximum pressure acting on the wall.

An inspection of Fig. 4 will show that the results for the pressure fluctuations obtained by solving the problem of collapse of a bubble close to a wall without allowance for its motion toward it [1] do not coincide with the results for the pressure fluctuations when the problem is solved in our formulation.

It is clear from Fig. 4 that at a fixed value of the gas content a decrease in the distance of the bubble from the wall leads to an increase in the pressure fluctuations at the wall, while at a fixed value of the initial distance of the bubble from the wall there are two regions of gas contents: in one, the region of low gas contents, the pressure is lower than for collapse of a cavitation bubble in an infinite medium.

NOTATION

R is the radius of cavity; $\dot{R}$ is the rate of collapse of the cavity; P_0 is the pressure in liquid in the cavity collapse zone, assumed constant during the collapse process; Q is the partial pressure of air in the cavity at the initial instant; P_d is the partial pressure of vapor in the liquid; ρ is the density of the liquid, φ is the potential of radial motion of the cavity; φ_Z, φ_R are the velocity potentials for unit velocity of translational motion and pulsation; T is the collapse period, $M = (r^2 + 4b^2 - 4br\cos\Theta)^{1/2}$; $N = (1 - \sin^2\Theta\,(r^2/M^2))^{1/2}$; b is the distance of the cavity from the boundary, U is the rate of motion of the cavity toward the boundary, $\dot{U} = dU/dt$.

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